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ON (z, x, y) SUMMABILITY OF JACOBI SERIES

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[1] 1. Let $\sum a_n$ be a given series with the sequence of partial sums $\{\delta_n\}$. Let $\{x_n\}, \{y_n\}$ be a sequence of constants, such that

$$\begin{aligned}(x^* y)_n &= x_0 y_n + x_1 y_{n-1} + \cdots x_n y_0 \\ &= \sum_{m=0}^{n} x_{n-m} y_m\end{aligned}$$

Let sequence to sequence transformation, be

$$t_n = \frac{1}{(x * y)_n} \sum_{m=0}^{n} x_{n-m} y_m \delta_m \tag{1.1}$$

The series $\sum_{m=0}^{\infty} a_m$ is said to be summable (z, x, y) to δ if $t_n \longrightarrow \delta$ as $n \longrightarrow \infty$. We shall denote it by

$$\sum_{m=0}^{\infty} a_m = \delta(z, x, y)$$

or

$$\delta_n \longrightarrow \delta(z, x, y).$$

We shall also use the notations

$$t_n = \frac{1}{D_n} \sum_{m=0}^{n} d_{n-m} y_v \delta_m$$

where

$$d_n = \Delta x_n = x_n - x_{n-m},$$

$$D_n = \sum_{m=0}^{n} \Delta x_m y_{n-m}.$$

[1.2] Let $f(\phi) = f(\cos\phi), \phi \in [0, \pi]$ be a Lebesgue measurable function such that

$$\int_0^{\pi} f(\phi) R_n^{(\alpha,\beta)}(\cos\phi) \sin\phi^{2\alpha+1} \cos\phi^{2\beta+1}\, d\phi$$

$$\alpha > -1, \beta > -1$$

exist, where $R_n{}^{(\alpha,\beta)}(\cos\phi)$ is the n^{th} Jacobi polynomial of order (α, β).

The Fourier Jacobi series associated with this function is

$$\hat{f}(n) \sim \int_0^{\pi} f(\phi)\, h_n R_n(\cos\phi) \qquad (1.2.1)$$

where

$$\hat{f}(n) \ = \int_0^{\pi} f(\phi) R_n(\cos\phi) d\mu(\phi)$$

and

$$h_n = \left\{ \int_0^{\pi} R_n \cos^2 d\mu(\phi) \right\}^{-1}$$

$$= \frac{\Gamma(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)}{(n+\beta+1)\Gamma(n+\alpha)\Gamma(\alpha+1)\Gamma(\beta+1)}$$

$$R_n(\cos\phi) = x_n^{(\alpha,\beta)}(\cos\phi)/x_n^{(\alpha,\beta)}(1)$$

and

$$d\mu(\phi) = (\sin\phi/2)^{2\alpha+1}(\cos\phi/2)^{2\beta+1}d\phi$$

ASKEY and WAINGER [1] have defined the convolution structure of two function f_1 and f_2 in the following manner.

$$(f_1^* f_2)\phi = \int f_1(\phi)\big(T_\theta f_2(\phi)\big)d\mu\phi$$

where the generalized translation T_θ is defined by

$$T_\theta f(\phi) = \int f(\psi)K(\phi,\theta,\psi)d\mu(\psi),$$

when

$$\int R_n(\cos\psi)K(\phi,\theta,\psi) = R_n(\cos\phi)R_n(\cos\psi)$$

and

$$\int K(\phi,\theta,\psi)d\mu = 1.$$

[1.3] Let

$$\delta_n(\phi) = \sum_{m=0}^{n} \hat{f}(m)h_m P_m(\cos\phi)$$

$$= \sum \int_0^\pi f(\theta) h_m P_m(\cos\phi) P_m(\cos\theta) d\mu\theta$$

Now using the orthogonal property, we obtain

$$\delta_n(\phi) - f(\phi) = \sum \{f(\theta) - f(\phi)\} d\mu(\theta) \times$$

$$\times \int_0^\pi \mathrm{K}(\phi, \theta, \mu) P_n(\cos\psi) d\mu(\psi)$$

$$= \int_0^\pi L_n\, \omega_f(\psi) R_n^{(\alpha+1,\beta)}(\cos\psi) d\mu\psi, \qquad (1.3.1)$$

where

$$\omega_f(\psi) = T_\psi(f) - f \qquad (1.3.2)$$

and

$$\mathrm{L_n} = \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} \sim n^{\alpha+1}$$

Therefore, we have

$T_n - f(\phi)$

$$= \frac{1}{(x^* y)_n} \sum_{k=0}^{n} \{\delta_{n+k}(\phi) - f(\phi)\} dk$$

$$= \frac{1}{(x^* y)_n} \sum_{k=0}^{n} d_k L_{n-k} \int_0^\pi \omega(\psi) P_{n-k}^{(\alpha+1,\beta)}(\cos\psi) d\psi$$

where

$$\omega(\psi) = \omega_f(\cos\psi)(\sin\psi/2)^{2\alpha+1}(\cos\psi/2)^{2\beta+1} \qquad (1.3.3)$$

INTRODUCTION

[1.4] In 1985 BEOHAR[2] proved the following theorem.

Theorem A : Let $\{x_n\}$ be a non-negative and nom-increating such that $\left\{x_n n^{-\left(\alpha+\frac{1}{2}\right)}\right\}$ is increasing.

if

$$\int_t^{\delta} \frac{|\,\omega(\psi)|P_{(1/\psi)}\big)d\psi}{\psi^{(\alpha+3)/2}} = O\big(P_{(1/t)}t^{(\alpha+1)/2}\big),$$

where

$$\psi = [1/t]$$

and

$$\int_0^{1/n} \left|\omega_f(\pi-\psi)\right|\psi^{\beta-\frac{1}{2}}d\psi = O(1) \qquad (1.4.1)$$

then for $\alpha > -\frac{1}{2}, \beta > -\frac{1}{2}$, the series

$$t_n = \frac{1}{R_n}\sum_{m=0}^{n} t_m\delta_{n-m}$$

is summable (z, x_n) to $f(\phi)$

Nörlund summability of the series $f(\phi) \sim \sum \hat{f}(n)h_nP_n(\cos\phi)$ at the end point has been studied by GUPTA [3] PANDEY AND BEOHAR [4] BEOHAR AND MISHRA [5] and ASKEY AND

WAINGER [6] the have applied the convolution structure formula and studied the series for the entire rang Q i.e. $[0, \pi]$.

In present paper we study the above theorem for (z, x, y) summability of the series (1.4.1) by applying the convolution structure formula.

What in follows, we prove the following:

1.5 THEOREM

Let $\{x_n\}$ and $\{y_n\}$ be non-negative and non-increasing such that $\left\{(x^*y)_n n^{-(\alpha+1/2)}\right\}$ is increasing.

If

$$\int_t^s \frac{|\omega(\psi)|\,(x^*y)\left(\frac{1}{\psi}\right)d\psi}{\psi^{\alpha+3/2}} = O\left((x^*y)_{\left(\frac{1}{t}\right)} t^{\alpha+\frac{1}{2}}\right. \quad (1.5.1)$$

$$\psi = \left[\frac{1}{t}\right]$$

and

$$\int_0^{\frac{1}{n}} \left|\omega_f(\pi-\psi)\right| \psi^{\beta-1/2} d\psi = 0(1). \quad (1.5.2)$$

then, for $\alpha > -\frac{1}{2}, \beta > -\frac{1}{2}$ the series $(1.2 \cdot 1)$ is summable (z, x, y) to $f(\phi)$.

[1.6] For the proof of the theorem, we require the following Lemmas.

LEMMA 1 : Let α, β be real and C a constant, then

$$Z_n(\psi) = \frac{1}{(x^*y)_n} \sum D_k L_{n-k} P_{n-k}^{(\alpha+1,\beta)}(cos\Psi).$$

$$Z_n(\psi) = O(n^{2\alpha+2}) \text{ for } 0 \leqslant \psi \leqslant \frac{1}{n} \quad (1.6.1)$$

$$Z_n(\psi) = O\left(\frac{n^{\frac{\alpha+1}{2}}}{(x^*y)_n}\right)\left[\frac{(x*y)_{\left[\frac{1}{t}\right]}}{(sin\frac{\Psi}{2})^{\alpha+\frac{3}{2}}(cos\frac{\Psi}{2})^{\beta+\frac{1}{2}}}\right]+$$

$$+O\left(n^{\frac{\alpha+1}{2}}\right)\left[\left(\sin\frac{\Psi}{2}\right)^{\frac{-\alpha-5}{2}}\left(\cos\frac{\Psi}{2}\right)^{\frac{-\beta-3}{2}}\right]$$

for $\frac{1}{n} \leqslant \psi \leqslant \pi^{\frac{1}{n}}$.

PROOF OF THE LEMMA 1:

We know that, $0 \leqslant \psi \leqslant \frac{1}{n}$

$$x_n^{\alpha+1,\beta}(\cos\psi) = O(n^{\alpha+1})$$

Therefore,

$$Z_n(\Psi) = 0\left(\frac{1}{(x^*y)_n}\right)\sum_{k=0}^{n}(n-k)^{\alpha+1}(n-k)^{\alpha+1}D_k$$

$$= 0\left(\frac{n^{2\alpha+2}}{(x^*y)_n}\sum_{k=0}^{n}D_k\right)$$

$$= 0(n^{2\alpha+2})$$

LEMMA 2: The condition (1.5.1), implies that

$$\int_0^t |\omega(\psi)|d\psi = O(t^{2\alpha+2}) \quad (1.6.3)$$

PROOF OF THE LEMMA 2:

Implies that condition (1.5.1)

$$\int_0^t |\omega(\psi)|(x^*y)_{\left[\frac{1}{\psi}\right]} = 0\left((x^*y)(t)_{1/t}t^{2\alpha+2}\right)$$

But the integral on the left hand side

$$\geqslant (x^*y)_{\left[\frac{1}{t}\right]}\int_0^t |\omega(\psi)|d\psi$$

Therefore, (1.6.3) is proved

1.7 PROOF OF THE THEOREM:

we have

$$t_n - f(\phi) = \int_0^\pi Z_n(\psi)\omega(\psi)d\psi$$

$$= \left\{\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\pi-\frac{1}{n}} + \int_{\pi-\frac{1}{n}}^{\pi}\right\} Z_n(\psi)\omega(\psi)d\psi$$

say $= A_1 + A_2 + A_3$

where

$$A_1 = \int_0^{\frac{1}{n}} O(n^{2\alpha+2})|\omega(\psi)|d\psi$$

by (1.6.1)

$$= \mathrm{O}(n^{2\alpha+2})\mathrm{O}(\psi^{2\alpha+2})_0^{\frac{1}{n}}$$

$$= O(1). \qquad (1.7.1)$$

Next, we consider

$$A_2 = \int_{\frac{1}{n}}^{\pi-\frac{1}{n}} \mathrm{O}\left(\frac{n^{\alpha+\frac{1}{2}}}{(x^*y)_n}\right)\left|(\omega(\psi))\right|(x^*y)_{\left[\frac{1}{\psi}\right]}\Bigg)/$$

$$/\left(\sin\frac{\psi}{2}\right)^{\alpha+3/2}\left(\cos\frac{\psi}{2}\right)^{\beta+\frac{1}{2}} \cdot d\psi +$$

$$+O(n^{\alpha-\frac{1}{2}}\int_{\frac{1}{n}}^{\pi-\frac{1}{n}} |\omega(\psi)\}| \left(\sin\frac{\psi}{2})^{\alpha-\frac{5}{2}}\left(\cos\frac{\psi}{2}\right)^{-\beta-\frac{3}{2}}\right)d\psi$$

$$= O\left(\frac{n^{\alpha+\frac{1}{2}}}{(x^*y)_n}\right)\int_{\frac{1}{n}}^{\pi-\frac{1}{n}} \frac{1\omega'(\psi)(x^*y)}{\psi^{\alpha+\frac{3}{2}}}\left(\frac{1}{\psi}\right)d\psi +$$

$$+o\left(n^{\alpha+\frac{1}{2}}\right)\int_{\frac{1}{n}}^{\pi-\frac{1}{n}} \frac{1\omega'(\psi)}{\psi^{\alpha+\frac{3}{2}}}d\psi.$$

$$= O(1) \text{ by } (1.5.1) \qquad (1.7.2)$$

Also, we get

$$A_3 = O(1) \text{ by } (1.5.2) \qquad (1.7.3)$$

Thus the theorem is proved.

Remark: For $y_n = 1$, our theerem reduces to BEOHAR [7]

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